Bicrossproduct structure of the null-plane quantum Poincaré algebra

1998 J. Phys. A: Math. Gen. 31 L1

(http://iopscience.iop.org/0305-4470/31/1/001)

J. Phys. A: Math. Gen. **31** (1998) L1–L7. Printed in the UK PII: S0305-4470(98)86273-2

LETTER TO THE EDITOR

Bicrossproduct structure of the null-plane quantum Poincaré algebra

Oscar Arratia†‖, Francisco J Herranz‡¶ and Mariano A del Olmo§+

† Departamento de Matemática Aplicada a la Ingeniería, Universidad de Valladolid. E-47011, Valladolid, Spain

‡ Departamento de Física, EU Politécnica, Universidad de Burgos, E-09006, Burgos, Spain

§ Departamento de Física Teórica, Universidad de Valladolid, E-47011, Valladolid, Spain

Received 24 July 1997, in final form 4 November 1997

Abstract. A nonlinear change of basis allows us to show that the non-standard quantum deformation of the $(3+1)$ Poincaré algebra has a bicrossproduct structure. Quantum universal R-matrix, Pauli–Lubanski and mass operators are presented in the new basis.

The aim of this letter is to prove that the non-standard quantum deformation of the $(3+1)$ Poincaré algebra [1], the so-called null-plane quantum Poincaré algebra, can be endowed with a structure of bicrossproduct Hopf algebra [2]. Such a structure was used some years ago by Majid [3] as an approach to physics at the Planck scale. The algebraic structure of the example worked out in [2, 3] is characterized by

$$[p, x] = 1 - \mathrm{e}^{-x} \qquad \Delta(x) = 1 \otimes x + x \otimes 1 \qquad \Delta(p) = p \otimes \mathrm{e}^{-x} + 1 \otimes p \tag{1}$$

and with triangular quantum R-matrix

$$R = \mathrm{e}^{x \otimes p} \mathrm{e}^{-p \otimes x}. \tag{2}$$

The null-plane quantum Poincaré algebra is one of the three known deformed Hopf structures supported by the Poincaré algebra. It is a triangular Hopf algebra whereas the κ-Poincaré [4–6] and the q-Poincaré [7] are quasitriangular ones. The null-plane formulation has a dynamical meaning, hence this scheme is not only relevant from a kinematical point of view. The quantum null-plane algebra was proposed for the study of deformed physical systems whose natural framework is the null-plane; for instance, systems in the infinite momentum frame approach, gauge-field theory on null-planes, hadronic spectroscopy, etc (see [1] and references therein).

After the proofs by Majid and Ruegg [8] that the $(3+1)$ κ-Poincaré algebra has a bicrossproduct structure, and more recently by Azcárraga *et al* [9] that the q-Poincaré in any dimension also has this kind of structure, it only remains to see if the same bicrossproduct structure is exhibited by the $(3+1)$ null-plane quantum Poincaré. In [10] it was shown that the $(1+1)$ null-plane quantum Poincaré [11] also shares this structure, however, this lower-dimensional case does not indicate the procedure for the $(3+1)$ case, i.e. the nonlinear

‖ E-mail address: oscarr@wmatem.eis.uva.es

¶ E-mail address: fteorica@cpd.uva.es

+ E-mail address: olmo@cpd.uva.es

change of basis that allows us to display the bicrossproduct structure. Note that in all the three mentioned deformations, the formal decomposition is the same, i.e.

$$U_q(\mathcal{P}(3+1)) = U(so(3,1))^{\beta}\triangleright\!\!\blacktriangleleft_{\alpha} U_q(\mathcal{T}_4)$$

following the same pattern of the classical algebra or group counterpart

$$P(3+1) = SO(3,1) \odot T_4$$

and with the sector of the translations deformed (differently in each case, of course) and the Lorentz transformation sector non-deformed.

On the other hand, an example of this kind of decomposition appeared some years ago in [12], $U(su(2))^{\beta}\triangleright\!\!\blacktriangleleft_{\alpha} C(\mathbb{R}^3)$ where $C(\mathbb{R}^3)$ is the Hopf algebra of functions defined on $\mathbb{R}^3$.

The generators of the (3+1) Poincaré algebra $\mathcal{P}(3+1)$ in the so-called null-plane basis [13] are

$$\{P_+, P_-, P_i, E_i, F_i, K_3, J_3; i = 1, 2\} \tag{3}$$

where P_+, P_-, E_i and F_i are expressed in terms of the usual kinematical ones $\{H, P_l, K_l, J_l; l = 1, 2, 3\}$ by

$$\begin{array}{lll} P_+ = (H + P_3)/2 & P_- = H - P_3 & E_1 = (K_1 + J_2)/2 \\ F_1 = K_1 - J_2 & F_2 = K_2 + J_1 & E_2 = (K_2 - J_1)/2. \end{array} \tag{4}$$

Hence, the Lie brackets of $\mathcal{P}(3+1)$ are (hereafter $i, j = 1, 2$):

$$\begin{array}{lll} [K_3, E_i] = E_i & [K_3, F_i] = -F_i & [K_3, J_3] = 0 \\ [J_3, E_i] = -\varepsilon_{ij3}E_j & [J_3, F_i] = -\varepsilon_{ij3}F_j & [E_1, E_2] = 0 \\ [E_i, F_j] = \delta_{ij}K_3 + \varepsilon_{ij3}J_3 & [F_1, F_2] = 0 & \end{array} \tag{5}$$

$$[P_\mu, P_\nu] = 0 \qquad \mu, \nu = +, -, 1, 2 \tag{6}$$

$$\begin{array}{lll} [K_3, P_+] = P_+ & [K_3, P_-] = -P_- & [K_3, P_i] = 0 \\ [J_3, P_i] = -\varepsilon_{ij3}P_j & [J_3, P_+] = 0 & [J_3, P_-] = 0 \\ [E_i, P_j] = \delta_{ij}P_+ & [E_i, P_-] = P_i & [E_i, P_+] = 0 \\ [F_i, P_j] = \delta_{ij}P_- & [F_i, P_+] = P_i & [F_i, P_-] = 0 \end{array} \tag{7}$$

where ε_{ijk} is the completely skew-symmetric tensor.

The semidirect product structure of the (3+1) Poincaré group, isomorphic to $ISO(3,1)$, can be clearly pointed out. The six generators $\{E_i, F_i, K_3, J_3\}$ close the Lorentz subgroup $SO(3,1)$ (5), while the four remaining $\{P_+, P_-, P_i\}$ generate the Abelian subgroup T_4 (6). Therefore, as is well known, $ISO(3,1) = SO(3,1) \odot T_4$.

A triangular or non-standard quantum deformation of $\mathcal{P}(3+1)$ was introduced in [1] in the null-plane framework mentioned above, whose Hopf structure we rewrite here for the sake of completeness and to clarify our main result. Let us denote the null-plane generators X displayed in (3), by $\tilde{X}$, and the deformation parameter by $\tilde{z}$.

Coproduct:

$$\begin{aligned}
\Delta(\tilde{X}) &= 1 \otimes \tilde{X} + \tilde{X} \otimes 1 \qquad \text{for } \tilde{X} \in \{\tilde{P}_+, \tilde{E}_i, \tilde{J}_3\} \\
\Delta(\tilde{Y}) &= \mathrm{e}^{-\tilde{z}\tilde{P}_+} \otimes \tilde{Y} + \tilde{Y} \otimes \mathrm{e}^{\tilde{z}\tilde{P}_+} \qquad \text{for } \tilde{Y} \in \{\tilde{P}_-, \tilde{P}_i\} \\
\Delta(\tilde{F}_1) &= \mathrm{e}^{-\tilde{z}\tilde{P}_+} \otimes \tilde{F}_1 + \tilde{F}_1 \otimes \mathrm{e}^{\tilde{z}\tilde{P}_+} + \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{E}_1 \otimes \tilde{P}_- - \tilde{z}\tilde{P}_- \otimes \tilde{E}_1\mathrm{e}^{\tilde{z}\tilde{P}_+} \\
&\quad + \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{J}_3 \otimes \tilde{P}_2 - \tilde{z}\tilde{P}_2 \otimes \tilde{J}_3\mathrm{e}^{\tilde{z}\tilde{P}_+} \\
\Delta(\tilde{F}_2) &= \mathrm{e}^{-\tilde{z}\tilde{P}_+} \otimes \tilde{F}_2 + \tilde{F}_2 \otimes \mathrm{e}^{\tilde{z}\tilde{P}_+} + \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{E}_2 \otimes \tilde{P}_- - \tilde{z}\tilde{P}_- \otimes \tilde{E}_2\mathrm{e}^{\tilde{z}\tilde{P}_+} \\
&\quad - \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{J}_3 \otimes \tilde{P}_1 + \tilde{z}\tilde{P}_1 \otimes \tilde{J}_3\mathrm{e}^{\tilde{z}\tilde{P}_+} \\
\Delta(\tilde{K}_3) &= \mathrm{e}^{-\tilde{z}\tilde{P}_+} \otimes \tilde{K}_3 + \tilde{K}_3 \otimes \mathrm{e}^{\tilde{z}\tilde{P}_+} + \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{E}_1 \otimes \tilde{P}_1 - \tilde{z}\tilde{P}_1 \otimes \tilde{E}_1\mathrm{e}^{\tilde{z}\tilde{P}_+} \\
&\quad + \tilde{z}\mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{E}_2 \otimes \tilde{P}_2 - \tilde{z}\tilde{P}_2 \otimes \tilde{E}_2\mathrm{e}^{\tilde{z}\tilde{P}_+}.
\end{aligned} \tag{8}$$

Co-unit and antipode:

$$\epsilon(\tilde{X}) = 0 \qquad \gamma(\tilde{X}) = -\mathrm{e}^{3\tilde{z}\tilde{P}_+}\tilde{X}\mathrm{e}^{-3\tilde{z}\tilde{P}_+} \qquad \text{for } \tilde{X} \in \{\tilde{P}_\pm, \tilde{P}_i, \tilde{E}_i, \tilde{F}_i, \tilde{K}_3, \tilde{J}_3\}. \tag{9}$$

Non-vanishing Lie brackets:

$$\begin{aligned}
&[\tilde{K}_3, \tilde{P}_+] = \frac{\sinh \tilde{z}\tilde{P}_+}{\tilde{z}} \qquad [\tilde{K}_3, \tilde{P}_-] = -\tilde{P}_- \cosh \tilde{z}\tilde{P}_+ \qquad [\tilde{K}_3, \tilde{E}_i] = \tilde{E}_i \cosh \tilde{z}\tilde{P}_+ \\
&[\tilde{K}_3, \tilde{F}_1] = -\tilde{F}_1 \cosh \tilde{z}\tilde{P}_+ + \tilde{z}\tilde{E}_1\tilde{P}_- \sinh \tilde{z}\tilde{P}_+ - \tilde{z}^2\tilde{P}_2\tilde{W}_+^z \\
&[\tilde{K}_3, \tilde{F}_2] = -\tilde{F}_2 \cosh \tilde{z}\tilde{P}_+ + \tilde{z}\tilde{E}_2\tilde{P}_- \sinh \tilde{z}\tilde{P}_+ + \tilde{z}^2\tilde{P}_1\tilde{W}_+^z \\
&[\tilde{J}_3, \tilde{P}_i] = -\varepsilon_{ij3}\tilde{P}_j \qquad [\tilde{J}_3, \tilde{E}_i] = -\varepsilon_{ij3}\tilde{E}_j \qquad [\tilde{J}_3, \tilde{F}_i] = -\varepsilon_{ij3}\tilde{F}_j \\
&[\tilde{E}_i, \tilde{P}_j] = \delta_{ij}\frac{\sinh \tilde{z}\tilde{P}_+}{\tilde{z}} \qquad [\tilde{F}_i, \tilde{P}_j] = \delta_{ij}\tilde{P}_- \cosh \tilde{z}\tilde{P}_+ \\
&[\tilde{E}_i, \tilde{F}_j] = \delta_{ij}\tilde{K}_3 + \varepsilon_{ij3}\tilde{J}_3 \cosh \tilde{z}\tilde{P}_+ \qquad [\tilde{P}_+, \tilde{F}_i] = -\tilde{P}_i \\
&[\tilde{F}_1, \tilde{F}_2] = \tilde{z}^2\tilde{P}_-\tilde{W}_+^z + \tilde{z}\tilde{P}_-\tilde{J}_3 \sinh \tilde{z}\tilde{P}_+ \qquad [\tilde{P}_-, \tilde{E}_i] = -\tilde{P}_i
\end{aligned} \tag{10}$$

where $\tilde{W}_+^z$ is a component of the deformed Pauli–Lubanski vector defined as

$$\tilde{W}_+^z = \tilde{E}_1\tilde{P}_2 - \tilde{E}_2\tilde{P}_1 + \tilde{J}_3\frac{\sinh \tilde{z}\tilde{P}_+}{\tilde{z}}. \tag{11}$$

In the following we show that this quantum algebra has a bicrossproduct structure [2]. Let us consider the map defined by:

$$\begin{aligned}
&P_+ = \tilde{P}_+ \qquad E_i = \tilde{E}_i \qquad J_3 = \tilde{J}_3 \qquad z = 2\tilde{z} \\
&P_- = \mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{P}_- \qquad P_i = \mathrm{e}^{-\tilde{z}\tilde{P}_+}\tilde{P}_i \\
&F_1 = \mathrm{e}^{-\tilde{z}\tilde{P}_+}(\tilde{F}_1 - \tilde{z}\tilde{E}_1\tilde{P}_- - \tilde{z}\tilde{J}_3\tilde{P}_2) \\
&F_2 = \mathrm{e}^{-\tilde{z}\tilde{P}_+}(\tilde{F}_2 - \tilde{z}\tilde{E}_2\tilde{P}_- + \tilde{z}\tilde{J}_3\tilde{P}_1) \\
&K_3 = \mathrm{e}^{-\tilde{z}\tilde{P}_+}(\tilde{K}_3 - \tilde{z}\tilde{E}_1\tilde{P}_1 - \tilde{z}\tilde{E}_2\tilde{P}_2).
\end{aligned} \tag{12}$$

By applying (12) to the Hopf algebra $U_{\tilde{z}}(\mathcal{P}(3+1))$, whose relations appear displayed in expressions (8)–(10), we obtain the Hopf algebra $U_z(\mathcal{P}(3+1))$, characterized by the following coproduct, co-unit, antipode and commutation relations:

$$
\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1 \qquad X \in \{P_+, E_i, J_3\} \\
&\Delta(Y) = \mathrm{e}^{-zP_+} \otimes Y + Y \otimes 1 \qquad Y \in \{P_-, P_i\} \\
&\Delta(F_1) = \mathrm{e}^{-zP_+} \otimes F_1 + F_1 \otimes 1 - zP_- \otimes E_1 - zP_2 \otimes J_3 \\
&\Delta(F_2) = \mathrm{e}^{-zP_+} \otimes F_2 + F_2 \otimes 1 - zP_- \otimes E_2 + zP_1 \otimes J_3 \\
&\Delta(K_3) = \mathrm{e}^{-zP_+} \otimes K_3 + K_3 \otimes 1 - zP_1 \otimes E_1 - zP_2 \otimes E_2
\end{aligned}
\tag{13}
$$

$$
\epsilon(X) = 0 \qquad X \in \{P_\pm, P_i, E_i, F_i, K_3, J_3\} \tag{14}
$$

$$
\begin{aligned}
&\gamma(X) = -X \qquad X \in \{P_+, E_i, J_3\} \\
&\gamma(Y) = -\mathrm{e}^{zP_+} Y \qquad Y \in \{P_-, P_i\} \\
&\gamma(F_1) = -\mathrm{e}^{zP_+}(F_1 + zP_- E_1 + zP_2 J_3) \\
&\gamma(F_2) = -\mathrm{e}^{zP_+}(F_2 + zP_- E_2 - zP_1 J_3) \;\; \gamma(K_3) = -\mathrm{e}^{zP_+}(K_3 + zP_1 E_1 + zP_2 E_2)
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&[K_3, E_i] = E_i \qquad [K_3, F_i] = -F_i \qquad [K_3, J_3] = 0 \\
&[J_3, E_i] = -\varepsilon_{ij3} E_j \qquad [J_3, F_i] = -\varepsilon_{ij3} F_j \qquad [E_1, E_2] = 0 \\
&[E_i, F_j] = \delta_{ij} K_3 + \varepsilon_{ij3} J_3 \qquad [F_1, F_2] = 0
\end{aligned}
\tag{16}
$$

$$
[P_\mu, P_\nu] = 0 \qquad \mu, \nu = +, -, 1, 2 \tag{17}
$$

$$
\begin{aligned}
&[K_3, P_+] = \frac{1 - \mathrm{e}^{-zP_+}}{z} \qquad [K_3, P_-] = -P_- - \frac{z}{2}(P_1^2 + P_2^2) \\
&[K_3, P_i] = (\mathrm{e}^{-zP_+} - 1) P_i \qquad [J_3, P_+] = 0 \qquad [J_3, P_-] = 0 \\
&[J_3, P_i] = -\varepsilon_{ij3} P_j \qquad [E_i, P_-] = P_i \qquad [E_i, P_+] = 0 \\
&[E_i, P_j] = \delta_{ij} \frac{1 - \mathrm{e}^{-zP_+}}{z} \qquad [F_i, P_+] = P_i \qquad [F_i, P_-] = -zP_i P_- \\
&[F_i, P_j] = -zP_i P_j + \delta_{ij}\left(\mathrm{e}^{-zP_+} P_- + \frac{z}{2}(P_1^2 + P_2^2)\right).
\end{aligned}
\tag{18}
$$

Note that the translation generators $\{P_+, P_-, P_i\}$ define a commutative but non-cocommutative Hopf subalgebra of $U_z(\mathcal{P}(3+1))$ denoted $U_z(\mathcal{T}_4)$, and the Lorentz sector is non-deformed at the algebra level.

Note the resemblance between the triangular Hopf algebra $\mathbb{R}^\beta \triangleright\!\!\blacktriangleleft_\alpha \mathbb{R}$, whose structure is displayed in (1), and the expressions for the generators K_3 and P_+. This similitude is more transparent in the $(1+1)$ null-plane quantum Poincaré algebra [10, 11].

Let us now consider the non-deformed Lorentz–Hopf algebra $U(so(3,1))$ spanned by the generators $\{E_i, F_i, K_3, J_3\}$ with classical commutation rules (5) and primitive coproduct: $\Delta(X) = 1 \otimes X + X \otimes 1$. We define a right action

$$
\alpha : U_z(\mathcal{T}_4) \otimes U(so(3,1)) \to U_z(\mathcal{T}_4) \tag{19}
$$

as

$$
\alpha(X \otimes Y) \equiv X \triangleleft Y := [X, Y] \qquad X \in \{P_\pm, P_i\} \qquad Y \in \{E_i, F_i, K_3, J_3\} \tag{20}
$$

explicitly

$$\begin{aligned}
&\alpha(P_+\otimes K_3)=\frac{\mathrm{e}^{-zP_+}-1}{z} \qquad \alpha(P_-\otimes K_3)=P_-+\frac{z}{2}(P_1^2+P_2^2)\\
&\alpha(P_i\otimes K_3)=(1-\mathrm{e}^{-zP_+})P_i \qquad \alpha(P_+\otimes J_3)=0 \qquad \alpha(P_-\otimes J_3)=0\\
&\alpha(P_i\otimes J_3)=\varepsilon_{ij3}P_j \qquad \alpha(P_-\otimes E_i)=-P_i \qquad \alpha(P_+\otimes E_i)=0\\
&\alpha(P_i\otimes E_j)=\delta_{ij}\frac{\mathrm{e}^{-zP_+}-1}{z} \qquad \alpha(P_+\otimes F_i)=-P_i \qquad \alpha(P_-\otimes F_i)=zP_iP_-\\
&\alpha(P_i\otimes F_j)=zP_iP_j-\delta_{ij}\left(\mathrm{e}^{-zP_+}P_-+\frac{z}{2}(P_1^2+P_2^2)\right).
\end{aligned} \tag{21}$$

The extension to the enveloping algebra is made taking into account the fact that

$$(ab)\triangleleft h=\sum(a\triangleleft h_{(1)})(b\triangleleft h_{(2)}) \qquad a\triangleleft(hk)=(a\triangleleft h)\triangleleft k \tag{22}$$

where

$$\Delta(h)=\sum h_{(1)}\otimes h_{(2)} \qquad a,b\in U_z(\mathcal{T}_4) \qquad \text{and} \qquad h,k\in U(so(3,1)).$$

Also we define a left coaction

$$\beta: U(so(3,1))\rightarrow U_z(\mathcal{T}_4)\otimes U(so(3,1)) \tag{23}$$

by

$$\begin{aligned}
&\beta(J_3)=1\otimes J_3 \qquad \beta(E_i)=1\otimes E_i\\
&\beta(F_1)=\mathrm{e}^{-zP_+}\otimes F_1-zP_-\otimes E_1-zP_2\otimes J_3\\
&\beta(F_2)=\mathrm{e}^{-zP_+}\otimes F_2-zP_-\otimes E_2+zP_1\otimes J_3\\
&\beta(K_3)=\mathrm{e}^{-zP_+}\otimes K_3-zP_1\otimes E_1-zP_2\otimes E_2
\end{aligned} \tag{24}$$

for the Lie generators of $so(3,1)$. In general, the coaction is not a homomorphism. The extension of the above definition to all the elements of $U(so(3,1))$ is made by means of the canonical projections $\pi: U_z(\mathcal{P}(3+1))\rightarrow U(so(3,1))$ and $p: U_z(\mathcal{P}(3+1))\rightarrow U_z(\mathcal{T}_4)$ using the expression

$$\beta(\pi(h))=\sum p(h_{(1)})\gamma(p(h_{(3)}))\otimes\pi(h_{(2)}) \qquad \forall h\in U(so(3,1)) \tag{25}$$

where $(1\otimes\Delta)\circ\Delta(h)=\sum h_{(1)}\otimes h_{(2)}\otimes h_{(3)}$, and γ is the antipode for the Hopf algebra $U_z(\mathcal{T}_4)$. Note that π is a Hopf algebra projection and p a co-algebra homomorphism [8, 14]. It can be shown that the right action α and left coaction β fulfil the compatibility conditions [2] in such a manner that $(U_z(\mathcal{T}_4),\alpha)$ is a right $U(so(3,1))$-module algebra and $(U(so(3,1)),\beta)$ is a left $U_z(\mathcal{T}_4)$-comodule co-algebra. We summarize the previous discussion in the following theorem, which is the main result of this letter together with the nonlinear basis change (12).

Theorem. The null-plane quantum Poincaré algebra has the bicrossproduct structure

$$U_z(\mathcal{P}(3+1))=U(so(3,1))^{\beta}\triangleright\!\!\blacktriangleleft_{\alpha}U_z(\mathcal{T}_4). \tag{26}$$

We would like to stress that the map (12) is invertible, so:

$$\begin{aligned}
&\tilde{P}_+=P_+ \qquad \tilde{E}_i=E_i \qquad \tilde{J}_3=J_3 \qquad \tilde{z}=z/2\\
&\tilde{P}_-=\mathrm{e}^{zP_+/2}P_- \qquad \tilde{P}_i=\mathrm{e}^{zP_+/2}P_i\\
&\tilde{F}_1=\mathrm{e}^{zP_+/2}(F_1+z(E_1P_-+J_3P_2)/2)\\
&\tilde{F}_2=\mathrm{e}^{zP_+/2}(F_2+z(E_2P_--J_3P_1)/2)\\
&\tilde{K}_3=\mathrm{e}^{zP_+/2}(K_3+z(E_1P_1+E_2P_2)/2).
\end{aligned} \tag{27}$$

This fact can be applied to reproduce in the bicrossproduct basis the physically relevant operators introduced in [1] such as Casimirs, spin, Hamiltonians and position operators. In particular, the deformed square of the mass M_z^2 is now

$$M_z^2 = 2P_- \frac{e^{zP_+} - 1}{z} - (P_1^2 + P_2^2)e^{zP_+} \tag{28}$$

and the square of the Pauli–Lubanski operator W_z^2 turns out to be

$$W_z^2 = (W_{13}^z)^2 + (W_{23}^z)^2 + \cosh(zP_+/2)(W_+^z W_-^z + W_-^z W_+^z) - z^2 M_z^2 (W_+^z)^2/4 \tag{29}$$

where

$$\begin{aligned} W_{i3}^z &= K_3 P_i e^{zP_+} + E_i P_- - F_i \frac{e^{zP_+} - 1}{z} + \frac{z}{2}(E_1 P_1 + E_2 P_2) P_i e^{zP_+} \\ &\quad + (-1)^i J_3 P_{3-i} \frac{e^{zP_+} - 1}{2} \qquad i = 1, 2 \\ W_-^z &= (F_1 P_2 - F_2 P_1) e^{zP_+} + J_3 P_- \frac{e^{zP_+} + 1}{2} + \frac{z}{2}(E_1 P_2 - E_2 P_1) P_- e^{zP_+} \\ &\quad + \frac{z}{2} J_3 (P_1^2 + P_2^2) e^{zP_+} \\ W_+^z &= (E_1 P_2 - E_2 P_1) e^{zP_+/2} + J_3 \frac{\sinh(zP_+/2)}{z/2}. \end{aligned} \tag{30}$$

The second-order Casimir (28) would give rise to a deformed Schrödinger equation in the same way as in [1, 15], while the Pauli–Lubanski vector (30) would allow us to derive quantum Hamiltonians and spin operators. However, it is clear that although the Hopf algebra structure of $U_z(\mathcal{P}(3+1))$ is rather simplified in this new basis, the associated operators adopt a much more complicated form than the original ones.

The map (12) resembles the one given in [16] which allowed us to deduce a (factorized) null-plane quantum universal R-matrix: both mappings are related by the interchange $e^{-\tilde{z}\tilde{P}_+} \leftrightarrow e^{\tilde{z}\tilde{P}_+}$. Hence, the universal R-matrix now reads

$$\begin{aligned} \mathcal{R} = \exp\{zE_2 \otimes e^{zP_+} P_2\} \exp\{zE_1 \otimes e^{zP_+} P_1\} \exp\{-zP_+ \otimes e^{zP_+} K_3\} \exp\{z e^{zP_+} K_3 \otimes P_+\} \\ \times \exp\{-z e^{zP_+} P_1 \otimes E_1\} \exp\{-z e^{zP_+} P_2 \otimes E_2\}. \end{aligned} \tag{31}$$

Therefore, each basis seems to be useful for a specific purpose and we do not find any privileged basis to express the whole quantum Poincaré algebra together with its associated elements (universal R-matrix, quantum Casimirs, etc).

In the particular case of $(1+1)$ dimensions [11] the triangular quantum R-matrix is analogous to (2). On the other hand, the R-matrix (31) restricted to the generators K_3 and P_+ is also in agreement with (2) up to a basis change. This fact together with the remark pointed after equation (18) show that, in some sense, the Hopf algebra introduced in [2, 3] lies on the basements of the algebraic structure of the null-plane Poincaré. This opens new possibilities for applying null-plane Poincaré to physics.

Finally to mention that the null-plane case in $(2+1)$ dimensions [15] also exhibits this bicrossproduct structure. It looks promising to use this bicrossproduct structure of the quantum algebras in order to study by duality the corresponding quantum groups. Work in this direction is in progress and will be published elsewhere.

This work was partially supported by the Dirección General de Enseñanza Superior (DGES grant PB95-0719) and DGICYT (project PB94-1115) from the Ministerio de Educación y Cultura of Spain.

References

[1] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1995 *Phys. Lett.* B **351** 137
[2] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[3] Majid S 1988 *Class. Quantum Grav.* **5** 1587
[4] Lukierski J, Nowicky A, Ruegg H and Tolstoy V 1991 *Phys. Lett.* B **264** 331
[5] Giller S, Koshiński P, Kunz J, Majewski M and Maślanka P 1992 *Phys. Lett.* B **286** 57
[6] Lukierski J, Nowicky A and Ruegg H 1992 *Phys. Lett.* B **293** 344
[7] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1994 *J. Math. Phys.* **35** 4928
Ballesteros A, Herranz F J, del Olmo M A and Santander M 1995 *Lett. Math. Phys.* **33** 273
[8] Majid S and Ruegg H 1994 *Phys. Lett.* B **334** 348
[9] de Azcárraga J A, del Olmo M A, Pérez Bueno J C and Santander M 1997 *J. Phys. A: Math. Gen.* **30** 3069
[10] de Azcárraga J A and Pérez Bueno J C 1996 *J. Phys. A: Math. Gen.* **29** 6353
[11] Ballesteros A, Herranz F J, del Olmo M A, Pereña C M and Santander M 1995 *J. Phys. A: Math. Gen.* **28** 7113
[12] Majid S 1991 *J. Funct. Anal.* **95** 291
[13] Leutwyler H and Stern J 1978 *Ann. Phys., NY* **112** 94
[14] Majid S 1990 *J. Algebra* **130** 17
[15] Ballesteros A and Herranz F J 1996 *Mod. Phys. Lett.* A **11** 1745
[16] Ballesteros A, Herranz F J and Pereña C M 1997 *Phys. Lett.* B **391** 71